Homework 7

5.1

2c $\begin{pmatrix} 0 & 1 & -1 \\ -1 & -1 & 0 \\ -1 & 0 & -2 \end{pmatrix}$ $T\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = (-1)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $T\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $T\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

Show that the three vectors are linearly independent

3d $\begin{vmatrix} 2-t & 0 & -1 \\ 4 & 1-t & -4 \\ 2 & 0 & -1-t \end{vmatrix} = (2-t)\begin{vmatrix} 1-t & -4 \\ 0 & -1-t \end{vmatrix} - \begin{vmatrix} 4 & 1-t \\ 2 & 0 \end{vmatrix} =$

$-(2-t)(1-t)(1+t) + 2(1-t) = (1-t)(t^2 - t - 2 + 2)$

$= -t(t-1)^2$ eigenvalues $\lambda = 0, \lambda = 1$

$\lambda = 0$ $\begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ $x = \frac{1}{2}z$, $y = 2z$

$z\begin{pmatrix} \frac{1}{2} \\ 2 \\ 1 \end{pmatrix}$. $\lambda = 1$ $\begin{pmatrix} 1 & 0 & -1 \\ 4 & 0 & -4 \\ 2 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x = z$ $\begin{pmatrix} x \\ y \\ x \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Basis of eigenvectors $\begin{pmatrix} \frac{1}{2} \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$Q = \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

8a 0 not an eigenvalue $\Leftrightarrow \forall v \neq 0, Tv \neq 0 \Leftrightarrow N(T) = \{0\}$ $\Leftrightarrow T$ is one-one $\Leftrightarrow T$ is invertible

(b) λ eigenvalue $\Leftrightarrow \exists v \neq 0, Tv = \lambda v \Leftrightarrow T^{-1}Tv = T^{-1}(\lambda v) \Leftrightarrow v = \lambda T^{-1}(v) \Leftrightarrow T^{-1}(v) = \lambda^{-1} v \Leftrightarrow$ λ^{-1} eigenvalue

14 char polynomial of $A = |A - tI| = |(A-tI)^t| = |A^t - (tI)^t|$
$= |A^t - tI|$ = char. polynomial of A^t

15(a) $Tx = \lambda x$ assume for induction that $T^{m-1}x = \lambda^{m-1}x$
$T^m x = T(T^{m-1}x) = T(\lambda^{m-1}x) = \lambda^{m-1}Tx = \lambda^{m-1}\lambda x =$
$\lambda^m x$

(b) By (a), $A^m x = L_{A^m}(x) = (L_A)^m x = \lambda^m x$

16(a) $B = Q^{-1}AQ$ $\quad Tr(B) = Tr((Q^{-1}A)Q) = Tr(Q(Q^{-1}A))$
$= Tr(A)$

(b) Given $T: V \to V$ let $A = [T]_\alpha$ for any bases α of V
Set $Tr\,T = Tr\,A$. The defenition is independent of choice of bases by (a).

5.2

3c $T(e_1) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, T(e_2) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, T(e_3) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

$f(t) = \begin{vmatrix} -t & 1 & 0 \\ -1 & -t & 0 \\ 0 & 0 & 2-t \end{vmatrix} = (t^2+1)(2-t)$ does not split

$\therefore$ not diagonalizable

3e $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}, T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}$ $f(t)$ splits by Fundamental Thm of Algebra.

$f(t) = |A - tI| = \begin{vmatrix} 1-t & i \\ i & 1-t \end{vmatrix} = (t-1)^2 - i^2 =$
$t^2 - 2t + 2$. Roots $\frac{2 \pm \sqrt{4-4\cdot 2}}{2} = 1 \pm i$

$\lambda_1 = 1+i$, $\lambda_2 = 1-i$ eigenvalues. Since $f(t)$ splits and the algebraic mult. of each eigenvalue is 1, alg. mult = geom. mult., so T is diagonalizable
A similar to D $\quad D = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$. Next find

eigenvector. $(A-\lambda_1 I)\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}-i & i\\ i & -i\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$\therefore$ $iu = iv$ so $u=v$ Basis for E_{λ_1} is $\begin{pmatrix}1\\1\end{pmatrix}$

Similarly $(A-\lambda_2 I)\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}i & i\\ i & i\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

so $u=-v$ and basis for E_{λ_2} is $\begin{pmatrix}1\\-1\end{pmatrix}$

$\therefore$ $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}$ is a basis of eigenvectors for T.

<u>11</u> A similar to T, where T is upper triangular matrix

$T = \begin{pmatrix}a_1 & & * \\ & a_2 & \\ 0 & & a_n\end{pmatrix}$ Then $f(t) = |T - tI| =$

$\det\begin{pmatrix}a_1-t & & * \\ & a_2-t & \\ 0 & & \ddots & \\ & & & a_n-t\end{pmatrix} = (-1)^n(t-a_1)(t-a_2)\cdots(t-a_n)$

$\therefore$ $a_1, \ldots, a_n$ are eigenvalues with possible reps. Write

$f(t) = (-1)^n(t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\cdots(t-\lambda_k)^{m_k}$.

The sequence of a's, $a_1, \ldots, a_n$ has the same members as the sequence $\underbrace{\lambda_1, \ldots, \lambda_1}_{m_1 \text{ times}}, \underbrace{\lambda_2, \ldots, \lambda_2}_{m_2 \text{ times}}, \ldots, \underbrace{\lambda_k, \ldots, \lambda_k}_{m_k \text{ times}}$

$\therefore$ $\operatorname{Tr} A = \operatorname{Tr} T = m_1\lambda_1 + \cdots + m_k\lambda_k$

$\det A = \det T = \lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_k^{m_k}$.

<u>12</u> (a) Write E_λ^T (resp. $E_{\lambda^{-1}}^{T^{-1}}$) for the eigenspace of T (resp. T^{-1}) corresponding to λ (resp. λ^{-1}).

$x \in E_\lambda^T \iff Tx = \lambda x \iff x = \lambda T^{-1}x \iff x \in E_{\lambda^{-1}}^{T^{-1}}$

(b) Let $A = [T]_\alpha$. Then $A^{-1} = [T^{-1}]_\alpha$. T diagonaliz. $\iff$ $A = Q^{-1}DQ \iff A^{-1} = Q^{-1}D^{-1}Q$ and $D^{-1} = \begin{pmatrix}a_1^{-1} & & 0\\ & \ddots & \\ 0 & & a_n^{-1}\end{pmatrix}$ diagonal $\iff T^{-1}$ diagonaliz.

$D = \begin{pmatrix}a_1 & & 0\\ & \ddots & \\ 0 & & a_n\end{pmatrix}$